CORE
Metadata, citation and similar papers at core.ac.uk
Provided by CERN Document Server

Universität Bonn
Physikalisches Institut

Low-Temperature Expansions and Correlation Functions of the $\mathbb{Z}_3$-Chiral Potts Model

by

N.S. Han[1]) and A. Honecker[2])

Abstract

Using perturbative methods we derive new results for the spectrum and correlation functions of the general $\mathbb{Z}_3$-chiral Potts quantum chain in the massive low-temperature phase. Explicit calculations of the ground state energy and the first excitations in the zero momentum sector give excellent approximations and confirm the general statement that the spectrum in the low-temperature phase of general $\mathbb{Z}_n$-spin quantum chains is identical to one in the high-temperature phase where the rôle of charge and boundary conditions are interchanged. Using a perturbative expansion of the ground state for the $\mathbb{Z}_3$ model we are able to gain some insight in correlation functions. We argue that they might be oscillating and give estimates for the oscillation length as well as the correlation length.

[1]) **Permanent address:** Department of Theoretical Physics, Hanoi State University, P.O. Box 600, Bo Ho, Hanoi, Vietnam

Post address:
Nußallee 12
W-5300 Bonn 1
Germany
[2]) e-mail:
unp06b@ibm.rhrz.uni-bonn.de

BONN-HE-93-13
hep-th/9304083
Bonn University
April 1993
ISSN-0172-8733

1. Introduction

The self-dual $\mathbb{Z}_3$-chiral Potts model was introduced by Howes, Kadanoff and den Nijs [1] and studied using e.g. fermionization and approximative methods, in particular perturbation expansions. One remarkable result of the perturbation expansions was that for special values of the parameters the first translationally invariant excitation is linear in the inverse temperature λ for special values of the parameters. v. Gehlen and Rittenberg then realized [2] that this model is integrable for these special values of the parameters because it satisfies the Dolan-Grady integrability condition [3] which is equivalent [4][5] to Onsager's algebra [6]. They also generalized this 'superintegrable' chiral Potts model to general $\mathbb{Z}_n$-spin n [2]. Afterwards, it attracted much attention because it can be related to a classical model that satisfies a generalized Yang-Baxter equation with Boltzmann weights defined on higher genus Riemannian surfaces [7 – 11]. However, even then perturbative methods lead to important new results [12][13]. One example is a conjecture for the exact form of the order parameters in general superintegrable $\mathbb{Z}_n$-chiral Potts chains [13].

Recently, a particle interpretation of the momentum zero sectors in the high-temperature phase of all $\mathbb{Z}_n$-chiral Potts models at general values of the parameters has been proposed [14] and a quasi-particle spectrum has been derived for the superintegrable $\mathbb{Z}_3$-chiral Potts model [15]. Furthermore, a scaling exponent for the wave vector in the low-temperature phase of the $\mathbb{Z}_3$-chiral Potts model has been calculated in [16][17] from level crossings in the ground state. This motivated us to perform the perturbative calculations reported in this paper. On the one hand the excitation spectrum in the low-temperature phase of general $\mathbb{Z}_n$-chiral Potts quantum chains is not completely understood. On the other hand very little is known about correlation functions. Perturbation expansions enable us to gain more insight in the spectrum of the low-temperature phase and shed some light on correlation functions.

A general $\mathbb{Z}_n$-spin quantum chain with N sites is defined by the Hamiltonian:

$$H_N^{(n)} = -\sum_{j=1}^{N}\sum_{k=1}^{n-1} \tilde{\lambda}\bar{\alpha}_k\sigma_j^k + \alpha_k\Gamma_j^k\Gamma_{j+1}^{n-k}. \tag{1.1}$$

For $n = 2$ (1.1) is just the well known Ising model. In this case the operators σ_j and Γ_j are the Pauli spin matrices σ_x and σ_z acting in a vector space $\mathbb{C}^2$ located at site j. For general n one may think of the operators σ_j and Γ_j as generalizations of the Pauli spin matrices – see (2.3) below. In this paper we identify the $N+1$st site with the 1st site, i.e. we use toroidal boundary conditions.

The Hamiltonians (1.1) contain $2n-1$ parameters. The temperature like parameter $\tilde{\lambda}$ will be chosen real while the coupling constants $\bar{\alpha}_k$ and α_k will be generally complex. $H_N^{(n)}$ is hermitian iff $\bar{\alpha}_k = \bar{\alpha}_{n-k}^*$ and $\alpha_k = \alpha_{n-k}^*$.

In this paper we will parametrize the constants $\bar{\alpha}_k$, α_k in (1.1) by two angles ϕ, φ, fixing their dependence on k:

$$\alpha_k = \frac{e^{i\phi(\frac{2k}{n}-1)}}{\sin\frac{\pi k}{n}}\ , \qquad \bar{\alpha}_k = \frac{e^{i\varphi(\frac{2k}{n}-1)}}{\sin\frac{\pi k}{n}}. \tag{1.2}$$

(1.2) is called the general 'chiral Potts model'. This parametrization is convenient because it can easily be specialized to various models. Setting $\phi = \varphi = 0$ yields models with a second order phase transition at $\lambda = 1$ that can be described by a parafermionic conformal field theory in the limit $N \to \infty$ [18]. These so-called Fateev-Zamolodchikov-models lead to extended conformal algebras $\mathcal{WA}_{n-1}$ where the simple fields have conformal dimension $2, \ldots, n$ [19][20]. The spectrum of the Hamiltonian (1.1) with $\phi = \varphi = 0$ can be described by the first unitary minimal model of the algebra $\mathcal{WA}_{n-1}$. For $n = 3$ it coincides with the three-states Potts model and the symmetry algebra is Zamolodchikov's well known spin-three extended conformal algebra [21] at $c = \frac{4}{5}$.

For $\phi = \varphi = \frac{\pi}{2}$ (1.2) specializes to the 'superintegrable' $\mathbb{Z}_n$-chiral Potts model which exhibits remarkable integrability properties [2]: At $\phi = \varphi = \frac{\pi}{2}$ the Hamiltonians (1.1) satisfy the Dolan-Grady integrability condition [3].

Albertini et al. [7−9] have shown that the Hamiltonian (1.1) can be obtained for more general values of the angles ϕ, φ as the τ-continuum limit of an integrable classical statistical model if one imposes the constraint

$$\tilde{\lambda} \cos \varphi = \cos \phi. \tag{1.3}$$

$H_N^{(n)}$ with the choices (1.2), (1.3) is in general no more self-dual. However, if we choose $\phi = \varphi$ in (1.2) $H_N^{(n)}$ is self-dual. Sometimes (1.3) is implied when referring to the chiral Potts model but we prefer to call (1.1) with (1.2) the general chiral Potts model.

The $\mathbb{Z}_3$ version of (1.1) is known to have four phases [8][22][17]: Two massive and two massless phases. One of the massive phases is ordered and the other massive phase is disordered. The low-temperature phase (small $\tilde{\lambda}$) that we are going to study in this paper is the ordered massive phase. At $\phi = \varphi = \frac{\pi}{2}$ it appears in the range $0 \leq \tilde{\lambda} \leq 0.901292\ldots$ [8][22].

In the next section we will review some well known facts about the Hamiltonian (1.1). Then, in section 3 we will evaluate the ground state energy and the first excitations perturbatively for $n = 3$ and zero momentum, but arbitrary ϕ, φ. The results remind us of a general duality statement that we discuss in section 4. Finally, section 5 is devoted to a study of correlation functions of the $\mathbb{Z}_3$-chain.

2. Preliminaries

In this section we summarize well known facts about $\mathbb{Z}_n$-quantum spin chains and introduce notations that will be useful later on. For a more detailed, recent review see e.g. [23].

First, we give a precise definition of the operators Γ_j and σ_j. σ_j and Γ_j freely generate a finite dimensional associative algebra by the following relations ($1 \leq j, l \leq N$):

$$\begin{aligned} \sigma_j \sigma_l &= \sigma_l \sigma_j \ , \qquad & \sigma_j \Gamma_l &= \Gamma_l \sigma_j \omega^{\delta_{j,l}} \ , \\ \Gamma_j \Gamma_l &= \Gamma_l \Gamma_j \ , \qquad & \sigma_j^n &= \Gamma_j^n = \mathbb{1} \end{aligned} \tag{2.1}$$

where ω is the nth root of unity $\omega = e^{\frac{2\pi i}{n}}$. In this paper we will only consider boundary conditions of type $\Gamma_{N+1} = \omega^{-R}\Gamma_1$, $R \in \mathbb{Z}_n$ for the Hamiltonian (1.1). We will mainly focus on periodic boundary conditions $\Gamma_{N+1} = \Gamma_1$.

The algebra (2.1) can be conveniently represented in $\otimes^N \mathbb{C}^n$. In this space we can choose the following basis if we label the standard basis of $\mathbb{C}^n$ by $\{e_0, \ldots, e_{n-1}\}$:

$$|\, i_1 \ldots i_N\rangle := e_{i_1} \otimes \ldots \otimes e_{i_N} \qquad 0 \leq i_j \leq n-1. \tag{2.2}$$

Usually one considers a special representation r of the algebra (2.1) – a definition can be found in appendix A. However, for low-temperature expansions of (1.1) it is more convenient to consider a different representation $\tilde{r}$:

$$\begin{aligned} \tilde{r}(\Gamma_j)\,|\, i_1 \ldots i_j \ldots i_N\rangle &= \omega^{i_j}\ |\, i_1 \ldots i_j \ldots i_N\rangle \\ \tilde{r}(\sigma_j)\,|\, i_1 \ldots i_j \ldots i_N\rangle &= |\, i_1 \ldots (i_j - 1 \bmod n) \ldots i_N\rangle\,. \end{aligned} \tag{2.3}$$

$H_N^{(n)}$ commutes with the $\mathbb{Z}_n$ charge operator $\hat{Q} = \prod_{j=1}^N \sigma_j$, thus has n charge sectors. The eigenvalues of $\hat{Q}$ have the form ω^Q with $Q \in \{0, \ldots, n-1\}$. We will refer to the number Q as the 'charge'.

$H_N^{(n)}$ also commutes with the translation operator T_N. The eigenvalues of T_N are Nth roots of unity. We label them by e^{iP} and call P the 'momentum'. For a chain of length N one has $P \in \{0, \frac{2\pi}{N}, \ldots, \frac{2\pi(N-1)}{N}\}$. The eigenstates $\|i_1 \ldots i_N\rangle\rangle_P$ with momentum P can be obtained from $|\, i_1 \ldots i_N\rangle$ by finite Fourier transformation. T_N acts for $R = 0$ on the states (2.2) as

$$\tilde{r}(T_N)\,|\, i_1 i_2 \ldots i_N\rangle = |\, i_2 \ldots i_N i_1\rangle\,. \tag{2.4}$$

In the case $R = 0$ the eigenstates with momentum P are given by

$$\|i_1 \ldots i_N\rangle\rangle_P := \frac{1}{\sqrt{\mathcal{N}}} \sum_{x=0}^{N-1} e^{iPx} \tilde{r}(T_N)^{-x}\ |\, i_1 \ldots i_N\rangle\,. \tag{2.5}$$

$\mathcal{N}$ is a suitable normalization constant. If the state $|\, i_1 \ldots i_N\rangle$ has no symmetry, one has $\mathcal{N} = N$. This will apply to all cases below where we need (2.5). For a definition of the momentum eigenstates (2.5) in case of boundary conditions $R \neq 0$ see e.g. [24][23]

3. Spectrum in the low-temperature phase

In this section we calculate the ground state energy and the lowest excitations in the low-temperature phase using perturbative expansions according to [25] around $\tilde{\lambda} = 0$ of the Hamiltonian (1.1). We restrict ourselves to periodic boundary conditions $R = 0$.

In [1] perturbation series were presented for the disorder operator (or magnetization m) and the first energy gap in the momentum zero sector of the superintegrable $\mathbb{Z}_3$-chiral Potts model leading to exact conjectures for both of them. After the superintegrable chiral Potts model had been generalized to general n [2] perturbation series for the ground state

energy, energy gap in the momentum zero sector, magnetization and susceptibility of this superintegrable $\mathbb{Z}_n$-chiral Potts model were presented in [12]. At the same time elaborate expansions of the ground state energy and some excitations of the superintegrable chiral Potts model for $n \in \{3, 4, 5\}$ and in particular perturbation series for the order parameters with general n were calculated in [13]. First perturbative results for the energy gaps at more general values of the angles ϕ, φ were obtained in [14] where second order high-temperature expansions for the translationally invariant energy gaps in each charge sector of the general self-dual $\mathbb{Z}_3$- and $\mathbb{Z}_4$-chiral Potts models as well as a first order expansion for the dispersion relations for general n was presented.

In this section we restrict once again to the $\mathbb{Z}_3$-version of the chiral Potts model (1.1) but impose no restrictions on the angles ϕ, φ. We present low-temperature expansions for the ground state energy and in particular the first translationally invariant energy gaps that up to now have not been treated by perturbative methods because of high degeneracies.

The normalization of the Hamiltonian (1.1) is chosen such that expansions around zero temperature $\tilde{\lambda} = 0$ are possible. If one wants to calculate expansions around infinite temperature one usually normalizes the Hamiltonian $\hat{H}_N^{(n)} = \frac{1}{\tilde{\lambda}} H_N^{(n)}$, sets $\lambda = \tilde{\lambda}^{-1}$ and performs expansions around $\lambda = 0$.

In each charge sector Q of the low-temperature phase there is one unique ground state. For arbitrary n it is given by:

$$|\,\mathrm{GS}; Q\rangle := \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \omega^{l \cdot Q}\, |\,l \ldots l\rangle \tag{3.1}$$

provided that $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$. For $n = 3$ (3.1) is the ground state if $-\pi \le \phi \le \pi$ and for $n = 4$ (3.1) is the ground state for $-\frac{5\pi}{6} \le \phi \le \frac{5\pi}{6}$. The excited states are more complicated and highly degenerate. The space of the first excitation is spanned by those states which have precisely two blocks of different spins. Furthermore, the values of the spins in these two blocks must have difference one. For fixed P, Q and $-\frac{\pi}{2} \le \phi \le \frac{\pi}{2}$ we can choose the following basis for the space of the first excitation:

$$|\,a_k^Q\rangle := \frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} \omega^{l \cdot Q} \| \underbrace{(l+1 \bmod n) \ldots (l+1 \bmod n)}_{k \text{ times}} l \ldots l \rangle\rangle_P. \tag{3.2}$$

In order to perform explicit calculations we now specialize to $n = 3$ with $P = 0$. The constant contribution in $\tilde{\lambda}$ to the ground state energy $E_{|\mathrm{GS};Q\rangle}$ and the first gap $\Delta E_{Q,1}$ can be calculated easily:

$$E^{(0)}_{|\mathrm{GS};Q\rangle} = -N \frac{4}{\sqrt{3}} \cos\left(\frac{\phi}{3}\right), \qquad \Delta E^{(0)}_{Q,1} = 4\sqrt{3} \cos\left(\frac{\phi}{3}\right). \tag{3.3}$$

As far as the ground state is concerned, we notice from the explicit form of the potential in (1.1) that the first order in the perturbation expansion vanishes. The next order that

has to be calculated is the second order $E^{(2)}_{|\mathrm{GS};Q\rangle}$. One applies the potential $V = \sum_{j,k} \bar{\alpha}_k \sigma_j^k$ to the ground state:

$$\tilde{r}(V)\,|\mathrm{GS};Q\rangle = -\frac{2\sqrt{N}}{\sqrt{3}}\left(e^{\frac{i\varphi}{3}}\,|a_1^Q\rangle + e^{-\frac{i\varphi}{3}}\omega^Q\,|a_{N-1}^Q\rangle\right). \tag{3.4}$$

Using $\Delta E^{(0)}_{Q,1} = E^{(0)}_{|a_k^Q\rangle} - E^{(0)}_{|\mathrm{GS};Q\rangle}$ one obtains the result

$$\begin{aligned} E^{(2)}_{|\mathrm{GS};Q\rangle} &= \sum_{|a\rangle \neq |\mathrm{GS};Q\rangle} \frac{|\,\langle a|\;\tilde{r}(V)\,|\mathrm{GS};Q\rangle\;|^2}{E^{(0)}_{|\mathrm{GS};Q\rangle} - E^{(0)}_{|a\rangle}} \\ &= -\frac{|\,\langle a_1^Q|\;\tilde{r}(V)\,|\mathrm{GS};Q\rangle\;|^2 + |\,\langle a_{N-1}^Q|\;\tilde{r}(V)\,|\mathrm{GS};Q\rangle\;|^2}{\Delta E^{(0)}_{Q,1}} \\ &= -\frac{2N}{3\sqrt{3}\cos\left(\frac{\phi}{3}\right)}. \end{aligned} \tag{3.5}$$

Set $\mathcal{C} := \cos\left(\frac{\phi}{3}\right)$, $\mathcal{R} := 1 - 4\mathcal{C}^2$. Then, calculating higher orders in the same manner, one arrives at:

$$\begin{aligned} Ne_0 := E_{|\mathrm{GS};Q\rangle}(N) = -N\Bigg(&\frac{4}{\sqrt{3}}\mathcal{C} + \frac{2}{3\sqrt{3}\mathcal{C}}\tilde{\lambda}^2 + \frac{\cos\varphi}{9\sqrt{3}\mathcal{C}^2}\tilde{\lambda}^3 - \frac{\sqrt{3}}{81\mathcal{C}}\left\{\frac{1}{2\mathcal{C}^2} + \frac{4}{\mathcal{R}}\right\}\tilde{\lambda}^4 \\ &- \frac{\sqrt{3}\cos\varphi}{81\mathcal{C}^2}\left\{\frac{3}{4\mathcal{C}^2} + \frac{4}{\mathcal{R}}\right\}\tilde{\lambda}^5\Bigg) + \mathcal{O}(\tilde{\lambda}^6). \end{aligned} \tag{3.6}$$

for sufficiently large N. For $\varphi = \phi = \frac{\pi}{2}$ (3.6) reproduces the result of [12]. The orders $e_0^{(k)}$ of the free energy per site e_0 are independent of N if $N > k$. This is a general feature of the ground state energy for spin quantum chains with nearest neighbour interaction.

Note that the expansion in powers of $\tilde{\lambda}$ (3.6) of e_0 does not depend on the charge sector for large N. From the explicit calculations we see that this is a general result: The order $\tilde{\lambda}^k e_0^{(k)}$ of the free energy does not depend on the charge if $N > k$. However, ground state level crossings have been observed in [16][17]. Indeed, for short chain length N high orders $\tilde{\lambda}^k$ $(k \geq N)$ do depend on the charge Q. Thus, at fixed $\tilde{\lambda}$ in the massive phase level crossings in the ground state do occur although the Q-dependent term of the ground state energy e_0 decreases fast in magnitude with increasing N. The presence of such level crosssings is a hint for oscillating correlation functions and, in fact, the critical exponent of the wave vector can be calculated from them [26][16][17]. For the superintegrable case $\phi = \varphi = \frac{\pi}{2}$ we explicitly determined level crossings in the ground state using 9th resp. 10th order expansions for $3 \leq N \leq 7$ sites. Our results for the temperatures $\tilde{\lambda}$ where the gap vanishes are in good agreement with the values presented in [17] (Table 3) and show in particular that there are no crossings besides the one presented in [17]. However, we have

argued that for larger N we would need even higher orders for the study of level crossings which goes beyond current computer power. Thus, numerical methods provide a much better tool for the study of level crossings [16][17].

Finally, we should mention that the approximation (3.6) is excellent for the whole massive low-temperature phase up to its boundary near $\tilde{\lambda} = 1$ already for moderately long chains. For example at the phase transition $\tilde{\lambda} = 1$ of the parity conserving Potts case $\phi = \varphi = 0$ one observes only a deviation of 0.8% between (3.6) and the exact result for $N = 12$ sites. In the superintegrable case $\phi = \varphi = \frac{\pi}{2}$ at $\tilde{\lambda} = 0.9$ (close to the boundary of the phase) the deviation of (3.6) at $N = 12$ from the exact result is also as small as 1.3%. Thus, the perturbation expansion (3.6) yields a surprisingly good approximation in the complete massive low-temperature phase. Even at the boundary of the phase the deviation is smaller than 2% for $N > 10$.

In principle, one can derive a critical exponent α for the specific heat $\frac{\mathrm{d}^2 e_0}{\mathrm{d}\tilde{\lambda}^2}$ from a perturbation expansion of the ground state energy e_0. However, (3.6) leads only to a third order expansion for the specific heat and on the superintegrable line even the third order vanishes. Although the approximation of (3.6) to the ground state energy e_0 itself is so good, one certainly needs higher orders for accurate estimates of the critical exponent of the specific heat. Consequently, α has been estimated using a 13th order expansion of e_0 in [1] for the self-dual case $\phi = \varphi = 0$ and for the superintegrable case $\phi = \varphi = \frac{\pi}{2}$ the Ising-like form of the eigenvalues has been exploited to calculate even higher orders of e_0 in [13]. The results in [1][13] indicate $\alpha = \frac{1}{3}$ independent of the angles ϕ, φ.

The calculation of the smallest gap $\Delta E_{Q,1}$ is more difficult. Let q be the projector onto the space spanned by the states $|\, a_k^Q \rangle$ (3.2). In this space, the potential acts as follows:

$$\begin{aligned}
q\tilde{r}(V)\,|\, a_1^Q\rangle &= -\frac{2}{\sqrt{3}}(2e^{\frac{i\varphi}{3}}\,|\, a_2^Q\rangle + e^{\frac{i\varphi}{3}}\omega^Q\,|\, a_{N-1}^Q\rangle\,) \\
q\tilde{r}(V)\,|\, a_k^Q\rangle &= -\frac{2}{\sqrt{3}}(2e^{\frac{i\varphi}{3}}\,|\, a_{k+1}^Q\rangle + 2e^{-\frac{i\varphi}{3}}\,|\, a_{k-1}^Q\rangle\,) \qquad 1 < k < N-1 \\
q\tilde{r}(V)\,|\, a_{N-1}^Q\rangle &= -\frac{2}{\sqrt{3}}(2e^{-\frac{i\varphi}{3}}\,|\, a_{N-2}^Q\rangle + e^{-\frac{i\varphi}{3}}\omega^{2Q}\,|\, a_1^Q\rangle\,).
\end{aligned} \tag{3.7}$$

In the limit $N \to \infty$ the lowest perturbative eigenvector converges to

$$\frac{1}{\sqrt{N-1}} \sum_{k=1}^{N-1} |\, a_k^Q\rangle\,. \tag{3.8}$$

Using (3.7) and (3.8) we can calculate that for $N \to \infty$ and $\phi < \frac{\pi}{2}$

$$\begin{aligned}
\lim_{N\to\infty} \Delta E_{Q,1} =& 4\sqrt{3}\mathcal{C} - \tilde{\lambda}\frac{8}{\sqrt{3}}\cos\left(\frac{\varphi}{3}\right) \\
&+ \tilde{\lambda}^2\left\{\frac{8\mathcal{C}\left(1+\cos\left(\frac{2\varphi}{3}\right)\right)}{3\sqrt{3}(\mathcal{R}+2)} + \frac{4\left(2-\cos\left(\frac{2\varphi}{3}\right)\right)}{3\sqrt{3}\mathcal{C}}\right\} + \mathcal{O}(\tilde{\lambda}^3).
\end{aligned} \tag{3.9}$$

Comparing (3.9) with the corresponding high-temperature expansion [14][27] shows that up to the order calculated it coincides with $\Delta E_{1,0}(\lambda)+\Delta E_{2,0}(\lambda)$ at the dual point $\lambda = \tilde{\lambda}\,\phi$, φ with interchanged. In fact, we can also estimate the error we have made in calculating (3.9). The error we are making when replacing the true eigenvector by (3.8) is of the magnitude N^{-1}. This behaviour is preserved when the potential V and the resolvent g are applied. Thus, we obtain in all orders starting with the first one a deviation which is of magnitude N^{-2}. This further supports the identification of (3.9) with the dual of a two-particle state. Of course, (3.9) holds only for $\phi < \frac{\pi}{2}$.

For $\phi \geq \frac{\pi}{2}$, the states (3.2) are not the first excited states any more. Now we have to consider the following states:

$$\begin{aligned}\frac{1}{\sqrt{n}}\Big(&\|1\ldots1\ 2\ldots2\ldots0\ldots0\rangle\rangle_{\,P}+\omega^{Q}\|2\ldots2\ldots1\ldots1\rangle\rangle_{\,P}\\ &+\ldots+\omega^{Q(n-1)}\|0\ldots0\ldots(n-1)\ldots(n-1)\rangle\rangle_{\,P}\Big).\end{aligned}\tag{3.10}$$

Going through the same steps as before we find:

$$\begin{aligned}\lim_{N\to\infty}\Delta E_{Q,1} =&12\sin\left(\frac{\pi-\phi}{3}\right)-\tilde{\lambda}4\sqrt{3}\cos\left(\frac{\varphi}{3}\right)\\ &-\tilde{\lambda}^2\frac{2}{\sqrt{3}}\left\{\frac{\cos\left(\frac{2\varphi}{3}\right)-2}{\mathcal{C}}+\frac{\cos\left(\frac{2\varphi}{3}\right)+1}{\cos\left(\frac{\pi-\phi}{3}\right)}\right\}+\mathcal{O}(\tilde{\lambda}^3),\qquad \phi\geq\frac{\pi}{2}.\end{aligned}\tag{3.11}$$

(3.11) coincides with $3\Delta E_{1,0}(\lambda)$ at the dual point $\lambda = \tilde{\lambda}$ with interchanged ϕ, φ in the corresponding high-temperature expansion [14][27] up to the order calculated.

For $\phi = \frac{\pi}{2}$ the states (3.2) and (3.10) are degenerate. However, for large N the dominant contribution comes from the states (3.10) such that (3.11) is valid for $\phi = \frac{\pi}{2}$ as well. At $\phi = \varphi = \frac{\pi}{2}$ this is in agreement with the exact result of [10]:

$$\lim_{N\to\infty}\Delta E_{Q,1} = 6(1-\tilde{\lambda})\qquad \text{for }\ \phi=\varphi=\frac{\pi}{2}.\tag{3.12}$$

4. Duality of spectra

The results in the previous section remind us of some well known results about duality: The spectra in the low-temperature phase at $\tilde{\lambda}$ are dual to those in the high-temperature phase at $\lambda = \tilde{\lambda}$ if we interchange α_k and $\bar{\alpha}_k$. This statement for $\mathbb{Z}_n$ quantum spin chains has been known for a long time [28][29] and was also used in [1]. However, special attention has to be paid to the boundary conditions when performing duality transformations. It has been observed in [2] that the duality transformation interchanges the rôle of the charge Q and boundary conditions R. In this section we reformulate the precise statement of duality for the general $\mathbb{Z}_n$-chiral Potts quantum chain (1.1) and discuss its consequences. For completeness, a simple non-standard proof is presented in appendix A.

We denote the Hamiltonian (1.1) including the parameters by $H_N^{(n)}(\tilde{\lambda}, R^{lt}, \bar{\alpha}_k^{lt}, \alpha_k^{lt})$. If the Hamiltonoperator is properly normalized for high-temperature expansions ($\hat{H}(\lambda) := \lambda H(\lambda^{-1})$) it will be called $\hat{H}_N^{(n)}(\lambda, R^{ht}, \bar{\alpha}_k^{ht}, \alpha_k^{ht})$, writing again explicitly the corresponding parameters. In order to be able to distinguish the parameters we have introduced an upper index. Furthermore, abbreviate the space with charge Q^{ht} in the high-temperature phase by $\mathcal{H}^{Q^{ht}}$ and the eigenspace of $\tilde{r}(\hat{Q})$ to eigenvalue $\omega^{Q^{lt}}$ by $\tilde{\mathcal{H}}^{Q^{lt}}$. Now we can formulate the statement: $\hat{H}_N^{(n)}(\lambda, R^{ht}, \bar{\alpha}_k^{ht}, \alpha_k^{ht})$ restricted to $\mathcal{H}^{Q^{ht}}$ and $H_N^{(n)}(\tilde{\lambda}, R^{lt}, \bar{\alpha}_k^{lt}, \alpha_k^{lt})$ restricted to $\tilde{\mathcal{H}}^{Q^{lt}}$ have the same spectra if

$$\begin{aligned} Q^{lt} &= R^{ht} \ , \qquad R^{lt} = Q^{ht} \ , \\ \bar{\alpha}_k^{lt} &= \alpha_k^{ht} \ , \qquad \alpha_k^{lt} = \bar{\alpha}_k^{ht} \ , \qquad \tilde{\lambda} = \lambda. \end{aligned} \tag{4.1}$$

The momentum decomposition can be applied alike in the high- and low-temperature phase. Thus, the statement of duality is also valid if we further restrict to eigenspaces with momentum P.

Note that the duality (4.1) preserves the condition (1.3). Thus, each integrable chiral Potts model is dual to exactly one integrable chiral Potts model. The integrable model is self-dual iff $\lambda = 1$ and $\phi = \varphi \in \frac{\pi}{2}\mathbb{Z}$. The point $\lambda = 1$, $\phi = \varphi = 0$ exhibits conformal invariance [18]. However, this is not true for the other self-dual points. For example the point $\lambda = 1$, $\phi = \varphi = \frac{\pi}{2}$ is not conformally invariant although in its vicinity non-diagonal conformal field theories have been used to derive correlation functions [30].

From (4.1) we conclude that the quasi-particle interpretation for the high-temperature phase of the general $\mathbb{Z}_n$-chiral Potts quantum chain [27] can be pulled over to the low-temperature phase. The duality transformation interchanges charge sector Q and boundary conditions R. Thus, the ground state of the high-temperature phase is mapped to periodic boundary conditions $R = 0$ in the low-temperature phase. However, the fundamental excitations are mapped to different boundary conditions corresponding to the charge sectors $R \in \{1, \ldots n-1\}$. Therefore we observed only composite particle states in section 3.

It is well known that in the high-temperature phase the limit $N \to \infty$ of the energy eigenvalues is independent of the boundary conditions whereas the charge is substantial. The duality (4.1) implies that in the low-temperature phase this limit does not depend on the charge, but the spectra are clearly different for different boundary conditions. The independence from the charge has already been observed in section 3 for some eigenvalues. One can also argue directly that this degeneracy holds for the complete spectra, at least in the range where the perturbation expansion converges.

In the perturbation expansion all orders can be organized with respect to the energy eigenvalues at zero temperature. All energy eigenspaces – apart from the ground state – have a dimension that grows at least with N. We have seen in section 3 for some examples that at a fixed order k of the perturbation expansion only a finite number $N_{E,k}$ of matrix elements of the potential V^k in the eigenspace to energy E depend on the charge Q. This holds generally as we can see from the proof of duality presented in appendix A (in particular (A.2) and (A.3)). Thus, the term for energy E at order k in the perturbation

expansion has a Q-dependent term that is at most of order $(\frac{N_{E,k}}{N})^2$ as $N \to \infty$. This implies that the differences between the charge sectors converge to zero in the large chain limit.

It remains to check that this is also true for the ground state. The contribution of the ground state to a perturbation expansion for any energy level other than for the ground state energy itself is neglegible. The kth order of the ground state energy per site $e_0^{(k)}$ is independent of the charge sector Q if $k < N$. This is easy to see because in the perturbation expansion application of k powers of the potential V to the ground state is projected back onto the ground state. Thus, all excitations that are created must be annihilated again, or an excited state is proportional to the ground state. The second case is only possible if an excitation is carried around the boundary of the chain implying $k \geq N$. On the other hand, the first possibility yields results that are clearly independent of the charge sector. Finally, convergence of the perturbation series for e_0 implies that the limit $N \to \infty$ is independent of the charge sector Q.

In summary, in the low-temperature regime all charge sectors are degenerate for $N \to \infty$, at least if the perturbation expansion converges.

5. Correlation functions in the low-temperature regime

In this section we will apply methods explained in more detail in [27] to the correlation functions in the low-temperature phase of the chiral $\mathbb{Z}_3$ Potts quantum chain. Note that the duality argument of section 4 applies only to the Hamiltonian and not to other operators. Thus, quantities like e.g. correlation lengths may be different in these two phases. In fact, current knowledge about the correlation functions is restricted to the conjectures presented in [22] and the argumentation in favour of a non-vanishing wave vector presented in [16][17].

We study the correlation functions

$$C_\Gamma(x) := \frac{\langle v|\ \Gamma_x^+ \Gamma_0\ |v\rangle}{\langle v|v\rangle}\ , \qquad C_\sigma(x) := \frac{\langle v|\ \sigma_x^+ \sigma_0\ |v\rangle}{\langle v|v\rangle} - \frac{\langle v|\ \sigma_x^+\ |v\rangle\, \langle v|\ \sigma_0\ |v\rangle}{\langle v|v\rangle^{\,2}}. \tag{5.1}$$

using a perturbative expansion [25] for the ground state $|v\rangle$ from the state $|\mathrm{GS}; Q\rangle$ (3.1). The expansion of the ground state in powers of $\tilde{\lambda}$ leads to an expansion of the correlation functions in powers of $\tilde{\lambda}$:

$$C_\Gamma(x) = \sum_{k=0}^{\infty} \tilde{\lambda}^k C_\Gamma^{(k)}(x)\ , \qquad C_\sigma(x) = \sum_{k=0}^{\infty} \tilde{\lambda}^k C_\sigma^{(k)}(x). \tag{5.2}$$

In the definition (5.1) it is legitimate to omit the one point functions for the operator Γ because they are zero due to charge conservation.

Below, we will first give the final results for general angles ϕ, φ. It turns out that the result is too complicated to infer the general form of the correlation functions (5.1). Thus, we then specialize to the superintegrable case $\phi = \varphi = \frac{\pi}{2}$ and calculate even higher orders. By looking for a good fit we try to guess the structure of the correlation functions. With

this experience we turn back to the general case and discuss how the correlation functions should change for general ϕ, φ.

In order to save space we present only the final results for the correlation functions. For $C_\Gamma(x)$ one obtains, using again the abbreviations $\mathcal{C} = \cos\left(\frac{\phi}{3}\right)$, $\mathcal{R} = 1 - 4\mathcal{C}^2$:

$$
\begin{aligned}
&C_\Gamma^{(0)}(x) = 1 \ , && C_\Gamma^{(1)}(x) = 0 \ , \\
&C_\Gamma^{(2)}(x) = \frac{1}{6\mathcal{C}^2}\left\{\delta_{x,0} - 1\right\} \ , && C_\Gamma^{(3)}(x) = \frac{\cos\varphi}{18\mathcal{C}^3}\left\{\delta_{x,0} - 1\right\} \ , \\
&C_\Gamma^{(4)}(x) = && \\
&\quad \frac{1}{27\mathcal{C}^2}\left\{(1-\delta_{x,0})\left(\frac{2(1-16\mathcal{C}^2)}{3\mathcal{R}^2} + \frac{1}{16\mathcal{C}^2}\right) + \delta_{x,1}\left(\frac{1 + 2\mathcal{C}^2 - 3\ i\ \sin\left(\frac{2\phi}{3}\right)}{3\mathcal{R}^2} + \frac{5}{16\mathcal{C}^2}\right)\right\}.
\end{aligned}
\tag{5.3}
$$

The first orders of the correlation function $C_\sigma(x)$ read as follows:

$$
\begin{aligned}
&C_\sigma^{(0)}(x) = \delta_{x,0} \ , && C_\sigma^{(1)}(x) = 0 \ , \\
&C_\sigma^{(2)}(x) = -\frac{1}{9}\left\{\frac{\delta_{x,0}}{\mathcal{C}^2} + \delta_{x,1}\left(\frac{2}{\mathcal{R}} + \frac{1}{2\mathcal{C}^2}\right)\right\} , && C_\sigma^{(3)}(x) = -\frac{\cos\varphi}{9\mathcal{C}}\left\{\frac{\delta_{x,0}}{2\mathcal{C}^2} + \frac{\delta_{x,1}}{\mathcal{R}}\right\} , \\
&C_\sigma^{(4)}(x) = && \\
&\frac{1}{81\mathcal{C}^2}\left\{\delta_{x,0}\left(\frac{7}{16\mathcal{C}^2} + \frac{8}{\mathcal{R}}\right) + \delta_{x,1}\left(\frac{15}{16\mathcal{C}^2} + \frac{29}{12\mathcal{R}} - \frac{16\mathcal{C}^4}{\mathcal{R}^3}\right) + \delta_{x,2}\left(-\frac{1}{8\mathcal{C}^2} + \frac{7 + 44\mathcal{C}^2}{12\mathcal{R}^2}\right)\right\}.
\end{aligned}
\tag{5.4}
$$

Note that the correlation functions (5.3) and (5.4) do not depend on the charge sector.

(5.3) suggests that the correlation function $C_\Gamma(x)$ tends to a non-zero constant for large distances x – in contrast to the correlation functions in the high-temperature phase [27] so that that the low-temperature phase is ordered over long ranges. However, beyond this general conclusion, it is difficult to guess from (5.3) or (5.4) what might be the behaviour even for small x. Thus, we set $\phi = \varphi = \frac{\pi}{2}$, calculate four further orders and obtain

$$
\begin{aligned}
&C_\Gamma^{(0)}(x) = 1 \ , && C_\Gamma^{(1)}(x) = C_\Gamma^{(3)}(x) = C_\Gamma^{(5)}(x) = C_\Gamma^{(7)}(x) = 0 \ , \\
&C_\Gamma^{(2)}(x) = \frac{2}{9}\left(\delta_{x,0} - 1\right) \ , && \\
&C_\Gamma^{(4)}(x) = \frac{7}{81}\left(\delta_{x,0} - 1\right) + \delta_{x,1}\frac{5 - i\sqrt{3}}{162} \ , && \\
&C_\Gamma^{(6)}(x) = \frac{1}{6561}\left\{336\left(\delta_{x,0} - 1\right) + 160\delta_{x,1} + 60\delta_{x,2}\right\} - i\frac{\sqrt{3}}{6561}\left\{26\delta_{x,1} + 20\delta_{x,2}\right\} \ , && \\
&C_\Gamma^{(8)}(x) = \frac{1}{354294}\left\{12600\left(\delta_{x,0} - 1\right) + 6852\delta_{x,1} + 3521\delta_{x,2} + 1225\delta_{x,3}\right\} && \\
&\qquad\qquad - i\frac{\sqrt{3}}{354294}\left\{960\delta_{x,1} + 995\delta_{x,2} + 525\delta_{x,3}\right\} &&
\end{aligned}
\tag{5.5}
$$

and

$$
\begin{aligned}
C_\sigma^{(0)}(x) =& \delta_{x,0} \ , \qquad\qquad C_\sigma^{(1)}(x) = C_\sigma^{(3)}(x) = C_\sigma^{(5)}(x) = C_\sigma^{(7)}(x) = 0 \ , \\
C_\sigma^{(2)}(x) =& \frac{1}{27} \left\{ -4\delta_{x,0} + \delta_{x,1} \right\} \ , \\
C_\sigma^{(4)}(x) =& \frac{1}{729} \left\{ -41\delta_{x,0} + 14\delta_{x,1} + 8\delta_{x,2} \right\} \ , \\
C_\sigma^{(6)}(x) =& \frac{1}{19683} \left\{ -586\delta_{x,0} + 147\delta_{x,1} + 126\delta_{x,2} + 80\delta_{x,3} \right\} \ , \\
C_\sigma^{(8)}(x) =& \frac{1}{531441} \left\{ -9927\delta_{x,0} + 2130\delta_{x,1} + 1721\delta_{x,2} + \kappa\delta_{x,3} + 910\delta_{x,4} \right\} .
\end{aligned}
\tag{5.6}
$$

Unfortunately, determination of the constant κ in $C_\sigma^{(8)}$ exceeded the numerical range of our special purpose computer algebra system.

Up to the order calculated, $C_\sigma(x)$ is real for all values of the parameters ϕ, φ, $\tilde{\lambda}$. $C_\Gamma(x)$, in contrast, has a non-vanishing imaginary part. By analogy to the high-temperature regime [27] and from the results in [16][17] one might expect that $C_\Gamma(x)$ is oscillating. Indeed, the correlation functions in the superintegrable case $\phi = \varphi = \frac{\pi}{2}$ can nicely be fitted by

$$C_\sigma(x) = a\delta_{x,0} + be^{-\frac{x}{\xi_\sigma}} \ , \tag{5.7a}$$

$$C_\Gamma(x) = m^2 + pe^{-\left(\frac{1}{\xi_\Gamma} + \frac{2\pi i}{L}\right)x} . \tag{5.7b}$$

For $\tilde{\lambda} \in \left\{ \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \right\}$ good fits to (5.5) and (5.6) using (5.7) are given by the values in the following table:

$\tilde{\lambda}$	ξ_σ	a	b	ξ_Γ	m^2	L	$\sin\left(\frac{2\pi}{L}\right)$	p
0.25	0.26(1)	0.89(1)	0.10(1)	0.24(3)	0.9857605	30 ± 18	0.26(3)	0.011(3)
0.50	0.41(3)	0.85(1)	0.11(1)	0.37(5)	0.9381	30 ± 15	0.27(1)	0.05(1)
0.75	0.59(9)	0.75(2)	0.14(2)	0.50(7)	0.832	30 ± 13	0.27(1)	0.15(2)

Table 1: Parameters for the correlation functions (5.7) at $\phi = \varphi = \frac{\pi}{2}$

First, we remark that the correlation lengths satisfy $\xi_\sigma = \xi_\Gamma =: \xi$ for all values of $\tilde{\lambda}$ within the numerical accuracy. In fact, one expects this equality because the correlation lengths should be the inverses of some mass scale, and there is only one mass scale in our problem because all three charge sectors are degenerate. Furthermore, we observe that our data is compatible with an oscillating correlation function for the operator Γ. The oscillation length L (or wave vector) is around 30 sites in a major part of the low-temperature phase. In [16][17] it has been predicted that L should diverge as $\tilde{\lambda}$ crosses the phase boundary and approaches $\tilde{\lambda} = 1$ where the critical exponent is expected to equal $\frac{2}{3}$. Our results are compatible with a divergent oscillation length at $\tilde{\lambda} = 1$ although due to the large errors we do not even see that L increases with $\tilde{\lambda}$.

We should mention that the linear approximation $e^{-\frac{2\pi i}{L}x} \approx 1 - ix \sin\left(\frac{2\pi}{L}\right)$ is as good as (5.7b). However, (5.7b) seems to be a more natural form. The fact that the relative error

of the estimate for q is much smaller than that of L just comes from the fact that L and q are related by exponentiation.

Note that the short correlation lengths strongly damp the correlation functions: At $x = 7$ where we expect the first zero of $C_\Gamma(x)$ it has already decreased by at least six orders of magnitude for $\tilde{\lambda} \leq \frac{3}{4}$. Closer to $\tilde{\lambda} = 1$ the correlation length should increase but so should the oscillation length as well. Thus, it will be very difficult to obtain more precise results from approximative arguments and an exact expression for $C_\Gamma(x)$ is probably needed in order to decide whether (5.7b) really is the correct form and to determine the wave vector L accurately.

Before we conclude the discussion of the correlation functions for the superintegrable chiral Potts model, we mention that a conjecture for the form of $C_\Gamma(x)$ has been formulated in [22]: $C_\Gamma(x) = m^2 + \mathcal{O}(e^{-\frac{x}{\xi_\Gamma}})$ where m is the order parameter. Our result (5.7) is compatible with this from. In [1] the conjecture for the order parameter

$$m = \frac{\langle \tilde{v} |\ \Gamma_x\ | \tilde{v} \rangle}{\langle \tilde{v} \,|\, \tilde{v} \rangle} = \left(1 - \tilde{\lambda}^2\right)^{\frac{1}{9}} \tag{5.8}$$

has been formulated, but (5.8) has not been proven yet. The constant term in (5.5) is in exact agreement with (5.7) and (5.8) up to the order calculated, such that we may assume that at least the constant term of $C_\Gamma(x)$ is now known exactly.

A few remarks on the choice of ground state (3.1) are in place because in [1][9] (5.8) has actually been derived considering an expectation value of the operator Γ_x. We have already pointed out that the one point functions of Γ_x vanish identically due to charge conservation if one uses the charge eigenstates (3.1). However, if one uses instead non-charge eigenstates like $|\, 0 \ldots 0 \rangle$ for a perturbative expansion of $|\, \tilde{v} \rangle$ they do not vanish. Indeed, using an expansion for $|\, \tilde{v} \rangle$ from $|\, 0 \ldots 0 \rangle$ we once again verified equality of this one point function with the order parameter m. If we redefine $\tilde{C}_\Gamma(x)$ by replacing $|\, v \rangle$ by $|\, \tilde{v} \rangle$ and subtracting the contribution from the one point functions, this is in fact the only change, i.e. $\tilde{C}_\Gamma(x) = C_\Gamma(x) - m^2$. $C_\sigma(x)$ remains unchanged under this redefinition.

For more general values of the angles ϕ, φ one expects the correlation functions to be also of the form (5.7) – of course with different values of the parameters. We can see from the constant term m^2 of the correlation function $C_\Gamma(x)$ (5.5) that it will not be of the form (5.8) for general $\phi \neq \frac{\pi}{2} \neq \varphi$. In general, the coefficient of $\tilde{\lambda}^3$ for the constant term does not vanish and m^2 does not even have an expansion in powers of $\tilde{\lambda}^2$. Among the powers that we have calculated for the general case only the fourth order in (5.5) has a non-vanishing imaginary part at $x = 1$. Under the assumption that (5.7b) is the general form we would expect the imaginary part at $x = 1$ to be proportional to $\sin\left(\frac{2\pi}{L}\right)$ for very small temperatures $\tilde{\lambda}$. Thus, we expect for very low temperatures $\tilde{\lambda}$ the relation $L^{-1} \sim \phi$. On the one hand, this explains the conjectured presence of a second length scale L in addition to the correlation length ξ. The oscillation length L just comes from the chiral angle ϕ and thus these two scales must be related to each other. On the other hand, the oscillation (should it really be present) will vanish smoothly as the parity conserving Potts case $\phi = \varphi = 0$ is approached.

6. Conclusion

In this paper we discussed the low-temperature phase of the $\mathbb{Z}_3$-chiral Potts quantum chain. We have perturbatively calculated the ground state energy and the first gaps for $P = 0$ at general chiral angles ϕ, φ. We explicitly observed duality to the high-temperature phase and independence of the charge sector Q. This demonstrates a general duality property stating equality of spectra in the low- and high-temperature phase. Thus, a quasi-particle interpretation for the high-temperature phase of the general $\mathbb{Z}_n$-chiral Potts quantum chain [27] can be pulled over to the low-temperature phase. However, charge Q and boundary conditions R are interchanged by the duality transformation. In particular, for periodic boundary conditions one sees only energy levels above the ground state that correspond to composite particle states. We also gave a general argument that in the infinite chain length limit all charge sectors are degenerate.

We have further studied correlation functions for the operators σ and Γ in the low-temperature phase of the $\mathbb{Z}_3$-chiral Potts quantum chain. The correlation function $C_\Gamma(x)$ has a constant term m^2 indicating long range order. Fitting (complex) exponential functions to the perturbation expansions of the correlation functions we estimated the correlation length ξ for $\phi = \varphi = \frac{\pi}{2}$ and found agreement with the prediction [16][17] that the correlation function C_Γ oscillates. A rough estimate for the oscillation length L has also been obtained. We argued that the oscillation length should satisfy $L \sim \phi^{-1}$ for small temperatures. In particular, the oscillation vanishes for vanishing chiral angle ϕ. These first results make an exact determination of the correlation functions desirable.

Acknowledgements

It is a pleasure to thank H. Hinrichsen and G.v. Gehlen for numerous valuable discussions.

One of us (N.S. H.) would like to thank the DAAD for financial support during his stay in Bonn when part of the work was done.

We are indebted to the Max-Planck-Institut für Mathematik in Bonn-Beuel for providing the computer time necessary for performing the more complicated algebraic calculations reported in this paper.

Appendix A: A Proof of duality

Duality has been proved in e.g. [28][29]. Still, we would like to present a slightly different approach in this appendix. We derive duality by comparing the representation $\tilde{r}$ (2.3) to the following representation r that is usually considered:

$$\begin{aligned} r(\sigma_j)\,|\,i_1\dots i_j\dots i_N\rangle &= \omega^{i_j}\,|\,i_1\dots i_j\dots i_N\rangle \\ r(\Gamma_j)\,|\,i_1\dots i_j\dots i_N\rangle &= |\,i_1\dots(i_j+1 \bmod n)\dots i_N\rangle\,. \end{aligned} \tag{A.1}$$

Note that $\tilde{r}(\Gamma_j) = r(\sigma_j)$ and $\tilde{r}(\sigma_j) = r(\Gamma_j^+)$ and that the representations $\tilde{r}$ and r are unitarily equivalent.

We recall the statement of section 4 before proving it. Let $\hat{H}_N^{(n)}(\lambda, R^{ht}, \bar{\alpha}_k^{ht}, \alpha_k^{ht})$ be the Hamiltonian with suitable normalization for high-temperature expansions ($\hat{H}(\lambda) := \lambda H(\lambda^{-1})$) and $H_N^{(n)}(\tilde{\lambda}, R^{lt}, \bar{\alpha}_k^{lt}, \alpha_k^{lt})$ be the Hamiltonian (1.1) with corresponding parameters. Furthermore, abbreviate the eigenspace of $r(\hat{Q})$ to eigenvalue $\omega^{Q^{ht}}$ by $\mathcal{H}^{Q^{ht}}$ and that of $\tilde{r}(\hat{Q})$ to eigenvalue $\omega^{Q^{lt}}$ by $\tilde{\mathcal{H}}^{Q^{lt}}$. Then $\hat{H}_N^{(n)}(\lambda, R^{ht}, \bar{\alpha}_k^{ht}, \alpha_k^{ht})$ restricted to $\mathcal{H}^{Q^{ht}}$ and $H_N^{(n)}(\tilde{\lambda}, R^{lt}, \bar{\alpha}_k^{lt}, \alpha_k^{lt})$ restricted to $\tilde{\mathcal{H}}^{Q^{lt}}$ have the same spectra if

$$\begin{aligned} &Q^{lt} = R^{ht}\,, \qquad R^{lt} = Q^{ht}\,, \\ &\bar{\alpha}_k^{lt} = \alpha_k^{ht}\,, \qquad \alpha_k^{lt} = \bar{\alpha}_k^{ht}\,, \qquad \tilde{\lambda} = \lambda. \end{aligned} \tag{4.1}$$

For the proof we fix the state $|\,\mathrm{GS}; Q^{lt}\rangle$ to be the ground state (3.1) in $\tilde{\mathcal{H}}^{Q^{lt}}$. Then the following states are a basis for $\tilde{\mathcal{H}}^{Q^{lt}}$ [1]):

$$|\,Q; i_2\dots i_N\rangle := \tilde{r}(\sigma_2^{-i_2})\dots\tilde{r}(\sigma_N^{-i_N})\,|\,\mathrm{GS}; Q^{lt}\rangle\,. \tag{A.2}$$

Note that this implies:

$$\tilde{r}(\sigma_1)\,|\,Q; i_2\dots i_N\rangle = \omega^{Q^{lt}}\,|\,Q; (i_2+1)\dots(i_N+1)\rangle\,. \tag{A.3}$$

Now consider the following intertwining isomorphism I:

$$I\,|\,Q; i_2\dots i_N\rangle := |\,(-i_2)(i_2-i_3)\dots(i_{N-1}-i_N)(i_N+R^{lt})\rangle\,. \tag{A.4}$$

Note that the map (A.4) maps the ground states in both phases onto each other. It is now straightforward to check using the basis (A.2) that

$$I\tilde{r}(\sigma_{(j+1 \bmod N)}) = r(\Gamma_j\Gamma_{j+1}^+)I\,, \qquad I\tilde{r}(\Gamma_j\Gamma_{j+1}^+) = r(\sigma_j)I\,. \tag{A.5}$$

The observation that I is a unitary map and r and $\tilde{r}$ are unitarily equivalent in conjunction with (A.5) proves duality.

[1]) these may be different from e.g. (3.2) and (3.10)

References

[1] S. Howes, L.P. Kadanoff, M. denNijs
Quantum Model for Commensurate-Incommensurate Transitions
Nucl. Phys. **B215** (1983) p. 169

[2] G.v. Gehlen, V. Rittenberg, *$\mathbb{Z}_n$-Symmetric Quantum Chains with an Infinite Set of Conserved Charges and $\mathbb{Z}_n$ Zero Modes*, Nucl. Phys. **B257** (1985) p. 351

[3] L. Dolan, M. Grady, *Conserved Charges from Self-Duality*
Phys. Rev. **D25** (1982) p. 1587

[4] B. Davies, *Onsager's Algebra and Superintegrability*
Jour. Phys. A: Math. Gen. **23** (1990) p. 2245

[5] B. Davies
Onsager's Algebra and the Dolan-Grady Condition in the non-Self-Dual Case
J. Math. Phys **32** (1991) p. 2945

[6] L. Onsager
Crystal Statistics. I. A Two-Dimensional Model with an Order-Disorder Transition
Physical Review **65** (1944) p. 117

[7] G. Albertini, B.M. McCoy, J.H.H. Perk, *Commensurate-Incommensurate Transition in the Ground State of the Superintegrable Chiral Potts Model*
Phys. Lett. **135A** (1989) p. 159

[8] G. Albertini, B.M. McCoy, J.H.H. Perk, *Level Crossing Transitions and the Massless Phases of the Superintegrable Chiral Potts Chain*, Phys. Lett. **139A** (1989) p. 204

[9] H. Au-Yang, B.M. McCoy, J.H.H. Perk, Sh. Tang, M.L. Yan
Commuting Transfer Matrices in the Chiral Potts Models: Solutions of Star-Triangle Equations with Genus > 1
Phys. Lett. **123A** (1987) p. 219

[10] R.J. Baxter, *The Superintegrable Chiral Potts Model*
Phys. Lett. **133A** (1988) p. 185

[11] B.M. McCoy, *The Chiral Potts Model: from Physics to Mathemityatics and back*
Special functions ICM 90, Satellite Conf. Proc., ed. M. Kashiwara and T. Miwa, Springer (1991) p. 245

[12] M. Henkel, J. Lacki
Integrable Chiral $\mathbb{Z}_n$ Quantum Chains and a New Class of Trigonometric Sums
Phys. Lett. **138A** (1989) p. 105

[13] G. Albertini, B.M. McCoy, J.H.H. Perk, S. Tang
Excitation Spectrum and Order Parameter for the Integrable N-State Chiral Potts Model
Nucl. Phys. **B314** (1989) p. 741

[14] G.v. Gehlen, A. Honecker, *Multi-Particle Structure in the $\mathbb{Z}_n$-Chiral Potts Models*
Jour. Phys. A: Math. Gen. **26** (1993) p. 1275

[15] S. Dasmahapatra, R. Kedem, B.M. McCoy
Spectrum and Completeness of the 3-State Superintegrable Chiral Potts Model
preprint ITB-SB 9211 (1992)

[16] G.v. Gehlen, T.W. Krallman, *Phase Structure of the $\mathbb{Z}_3$-Chiral Potts Model*
in preparation

[17] G.v. Gehlen, *Phase Diagram and Two-Particle Structure of the $\mathbb{Z}_3$-Chiral Potts Model*

preprint BONN-HE-92-18 (1992), hep-th/9207022, to be published in proceedings of *Int. Symposium on Advanced Topis of Quantum Physics*, Taiyuan, China (1992)

[18] F.C. Alcaraz, A.L. Santos, *Conservation Laws for $\mathbb{Z}(N)$ Symmetric Quantum Spin Models and Their Exact Ground State Energies* Nucl. Phys. **B275** (1986) p. 436

[19] V.A. Fateev, A.B. Zamolodchikov *Conformal Quantum Field Theory Models in Two Dimensions Having $\mathbb{Z}_3$ Symmetry*, Nucl. Phys. **B280** (1987) p. 644

[20] V.A. Fateev, S.L. Lykyanov, *The Models of Two-Dimensional Conformal Quantum Field Theory with $\mathbb{Z}_n$ Symmetry* Int. Jour. of Mod. Phys. **A3** (1988) p. 507

[21] A.B. Zamolodchikov *Infinite Additional Symmetries in Two-Dimensional Conformal Quantum Field Theory*, Theor. Math. Phys. 65 (1986) p. 1205

[22] G. Albertini, B.M. McCoy, J.H.H. Perk, *Eigenvalue Spectrum of the Superintegrable Chiral Potts Model*, Adv. Studies in Pure Math. **19** (1989) p. 1

[23] Ph. Christe, M. Henkel *Introduction to Conformal Invariance and its Applications to Critical Phenomena* preprint UGVA/DPT 1992/11 - 794, to appear in Springer Lecture Notes in Physics

[24] G.v. Gehlen, V. Rittenberg *The Ashkin-Teller Quantum Chain and Conformal Invariance* Jour. Phys. A: Math. Gen. **20** (1987) p. 227

[25] G. Baym, *Lectures on Quantum Mechanics* Benjamin/Cummings Publishing (1969), 3rd printing (1974), chapter 11

[26] C. Hoeger, G.v. Gehlen, V. Rittenberg *Finite-Size Scaling for Quantum Chains with an Oscillatory Energy Gap* Jour. Phys. A: Math. Gen. **18** (1985) p. 1813

[27] G.v. Gehlen, A. Honecker *Quasiparticle Spectrum and Correlation Functions of the Chiral Potts Model* in preparation

[28] S. Elitzur, R.B. Pearson, J. Shigemitsu *Phase Structure of Discrete Abelian Spin and Gauge Systems* Phys. Rev. **D19** (1979) p. 3698

[29] D. Horn, M. Weinstein, S. Yankielowicz *Hamiltonian Approach to $\mathbb{Z}(N)$ Lattice Gauge Theories* Phys. Rev. **D19** (1979) p. 3715

[30] G. Albertini, B.M. McCoy, *Correlation Functions of the Chiral Potts Chain from Conformal Field Theory and Finite-Size Corrections* Nucl. Phys. **B350** (1990) p. 745